

RELATIVISTIC CRYSTALLINE SYMMETRY BREAKING AND ANYONIC STATES IN MAGNETOELECTRIC SUPERCONDUCTORS*

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Abstract

There exists a connection between the creation of toroidal moments (TM) and the breaking of the one-cell relativistic crystalline symmetry (RCS) associated to any given crystal [1] into which non-trivial magnetoelectric coupling effects (ME) exist [2]. Indeed, in this kind of crystals, any interaction between a charge carrier and an elementary magnetic cell can breaks the RCS of this previous given cell by varying, in the simplest case, the continuous defining parameters of the initial RCS.

This breaking can be associated to a change of the initial Galilean proper frame of any given carrier to an “effective” one, into which the RCS of the interacting cell is kept. We can speak of a kind of “inverse” kineto-magnetoelectric effect [3]. The magnetic groups compatible with such process have been computed [1]. Moreover, one can notice that the TM’s break the P and T symmetries but not the PT one as in anyons theories [4, 5]. This breaking creates so-called Nambu–Goldstone bosons generating “effective” magnetic monopoles. These consequences allow us to claim, first, that anyons are

charge carriers associated with “effective” magnetic monopoles, both with TM’s, and second, that ME can be highly considered in superconductors theory.

KEY WORDS: *toroidal moments, relativistic crystalline symmetries, symmetry breaking, anyons, Chern-Simon Lagrangians.*

1 - Quantized Hall effect, high- T_c superconductivity and anyons theory

In this chapter, we very briefly recall the quantized Hall effect and the link with both the high- T_c superconductivity theory and the anyons theory [4]. It is well-known that applying a magnetic field in the orthogonal direction of a conducting layer crossed over by an electric current, then an electric field orthogonal both to the longitudinal electric current and the magnetic field appears. That is the so-called classical Hall effect. In addition, a magneto-resistivity (or magneto-conductivity) effect changes, in particularly, values of the longitudinal resistivity.

Then, currents and electric fields are no longer proportionnal but are related together by a two-dimensional antisymmetric resistivity tensor depending on the applied magnetic field inten-

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sity B . Each diagonal coefficient equals the longitudinal resistance, whereas the non-diagonal coefficients are precisely plus or minus the so-called Hall resistance $R_H = B/(nec)$, where n is the surface density of electric charges. The latter is linearly depending on the filling factor $\nu = n/n_B$. This previous relation links the Hall resistance to the number of filled so-called Landau levels in a bounded conducting layer. These levels are defined by the stationary eigenfunctions of a one-particle Schrödinger equation in a two-dimensional space with a magnetic interaction. It is a harmonic oscillator-like equation with eigenvalues $E_N = \hbar\omega_C(N+1/2)$, where ω_C is the cyclotron frequency ($N \in \mathbb{N}$). The boundary conditions make each Landau level highly degenerate with a n_B surface density of degeneracies.

So, a linear variation of the Hall resistance would have been experimentally observed, but as K. Von Klitzing and al. [6] have shown in 1980, the Hall resistance is quantized and the evolution with the magnetic intensity presents plateaus for integer values of ν . More, on the plateaus and when the temperature goes to zero, the longitudinal resistance tends also towards zero. The system becomes non-dissipative and permanent longitudinal currents appear.

In 1982, D.C. Tsui and al. [7] observed the fractional quantized Hall effect for rational values of $\nu \leq 1$ ($\nu = p/q$ with q odd), meaning that only the first Landau level is excited and that electron-electron correlations produce condensation in a lower energy level. Such results have been explained in 1983 by B. B. Laughlin [8] considering a model of free fermions in a plane, interacting with a very particular gauge potential (the “anyonic” vector potential) not a priori electromagnetic. The main characteristic of this potential is the existence of singularities in the plane at each locations of the fermions. Solving the corresponding Hamiltonian leads to the so-called Laughlin function, which points out strong correlations at a finite distance depending on the gauge potential intensity. Moreover, the computation of the total energy of the system effectively shows singularities (“cusps”) for rational values (with an odd denominator) of ν .

One can display this model in considering excited fractional electric charges e/q , i.e. each cyclotron orbit contains into its associated circle, an odd number q of singularities (or electrons) of the gauge potential; The latter being

viewed as effective magnetic singular flux tubes. The electric charge is shared by q of these flux tubes. It is from these considerations that the anyons theory and the anyonic superconductivity are built. Indeed, the gauge choice made by Laughlin, generates a non-trivial Aharonov-Böhm phase when each electron moves on its cyclotron orbit. In particularly, it follows that the permutation or exchange operator has no longer the eigenvalues ± 1 only, but more generally, unitary complex numbers. That is the main property of anyons together with the P and T violations.

Actually, the litterature about anyons theory and its mathematical formalisms and tools, is highly increased. The basic mathematical tool of all of these formalisms, is to take into account the so-called Chern-Simon (CS) Lagrangian term [4, 9]. The latter breaks, as expected, the P and T symmetries and is associated to the singularities of the gauge fields. It is necessary to point out a common confusion about the CS term. It is absolutely not the CS term in a 2+1 dimension space (it doesn’t exist !), but the one of a curvature field in the 3+1 Minkowski space ! After integrating on a 2+1 dimensional hypersurface, bounded by the plane of the conducting layer, we obtain the “CS term” of the two-dimensional anyons theory. Moreover, the 3+1 CS term is ${}^*F \cdot F$, where F is the Faraday tensor and *F its Hodge dual. Hence, in this paper, we consider anyons theory in 3+1 dimensions and then the restriction onto the conducting layer.

Generally, this term is added up in order to consider monopoles such as the Dirac one. Let us recall that the Dirac monopole is associated with a singular gauge potential. Hence people have made models of anyons considering interaction between free electrons and magnetic monopoles. As S. Mandelstam showed [10], this leads to a confinement of the Fermi gas and may generate superconductor states. Anyway, statistical calculus based on the equivalence between anyons gas and free electrons embeded in a static magnetic field, show that the anyons gas generates a Meissner effect. That is the main reason, with the existence of permanent currents in the quantized Hall effect, for using anyons theory in order to explain high- T_c superconductivity in two dimensions.

2- Toroidal moments and broken relativistic crystalline symmetries

Electric and magnetic crystals are characterized by their magnetic groups which are subgroups of the so-called Shubnikov group $O(3)1'$ (the time inversion is indicated by the “'” symbol). Among the 122 magnetic groups, only 106 are compatible with the existence of a linear or quadratic magnetoelectric effect.

In the present paper, we consider relativistic symmetry group theory in crystals [11]. Therefore, we need, first, an extension from the Shubnikov group $O(3)1'$ to the group $O(1, 3)$ in the Minkowski space, and second and more particularly, transformations of $O(1, 3)$ leaving invariant polarization and magnetization vectors, and generating a subgroup of the normalizer $N(G)$ of G in $O(1, 3)$. This subgroup G' may not be identified with the magnetic group if G leaves invariant a particular non-vanishing velocity vector. If such a vector exists and $G' \neq G$, one strictly speaks about the relativistic crystalline symmetry G' . Only 31 magnetic groups are compatible with the existence of a relativistic crystalline symmetry.

The invariant non-vanishing velocity vectors can be linked with toroidal moments from the point of view of magnetic symmetries, as it has been already shown in previous papers (see [12] for instance). The toroidal moments \mathbf{T} are polar tensors which change sign under time inversion, like velocity vectors or current vectors of electric charges. There are referred to the order parameter in toroidal phase transitions, and involved, in particular, in the superdiamagnetism of superconductors or dielectric diamagnetic bodies containing densely packed atoms (aggregates) [12].

In such systems, in the presence of spontaneous currents, there may exist states for which the configuration of the associated currents has a torus-shaped solenoid with a winding and so a toroidal moment. For a system with a dipolar toroidal moment density \vec{T} , the current density equals

$$\vec{j} = \text{curl}(\text{curl}(\vec{T})). \quad (1)$$

We can remark on Figure 1 below

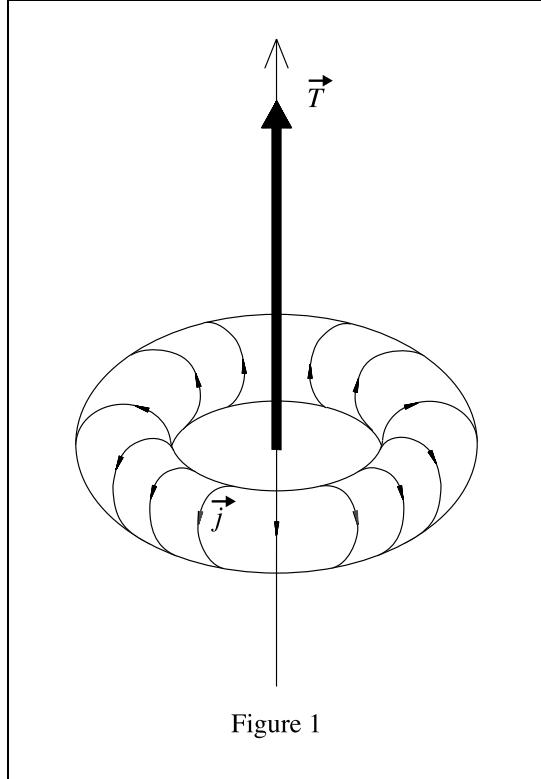


Figure 1

that this kind of configuration of currents can be generated from a solenoidal one. The creation or the closure to a torus-shaped solenoid from the latter, can be obtained adiabatically, applying for instance, a homogeneous external magnetic field slowly rotating at a more lower frequency than the cyclotron one. But we present another possibility in connection with the breaking of the relativistic crystalline symmetries. Before, it is absolutely necessary to understand that the groups G' are strongly depending on the orientations of the polarization and/or magnetization vectors.

Thus, if for example the system passes from the polarization vector $\vec{P} = \vec{0}$ to $\vec{P} \neq \vec{0}$, it involves a breaking, by group conjugaison, of the relativistic crystalline symmetry. Moreover, this conjugaison can be associated to a change of Galilean frame, from an initially fixed one to an “effective” moving one, in such a way that the initial G' group is kept in the latter.

Then, phenomenologically, we can consider for instance, the following classical physical system: a relativistic crystalline group G'_0 of an elementary polarized cell of a given crystal, with an electron not in interaction with this magnetic cell in the initial state. We also assume that this electron has initially a Galilean motion with, for instance, a velocity \vec{v} parallel to the invariant

axis of G'_0 (it is necessary to assume that this group is not invariant by smooth translations in order to have only one such axis). Then, the electron interacts with the cell, inducing (by a local magnetoelectric influence effect, by an exchange interaction for instance or in fact whatever is the interaction !) a change of the electric or magnetic polarization which breaks the initial relativistic crystalline symmetry group. During the interaction, the resulting symmetry group is G'_i in the fixed laboratory frame. The transformation from one symmetry to the other is realized, by group conjugaison, with an element S of $N(G)$. It means that we pass with S from one frame to another in which the motion of the electron is kept, i.e. Galilean. In some way, we can speak of a kind of “inverse kinetomagnetoelectric effect”, since usually the crystal moves whereas it is motionless in that present case.

Thus, the motion of the electron in the laboratory frame is determined by S , i.e. at most a rotation around the invariant axis and a boost along the same one. From the latter characteristics of the electron motion and those of S , the electron will get a solenoidal motion around the invariant axis, and if the electron is polarized, the Thomas precession of its spin along the trajectory would lead to a toroidal configuration of spin currents [13]. Moreover, if the electron has a fast solenoidal motion before the interaction with the cell, then with the condition that the interaction is adiabatic, one will obtain, by the latter process, a slow solenoidal motion of the mean position of the electron, and so a tore-shaped motion during the interaction. Let us recall also that a non trivial Berry phase can occur in this process with two main consequences: an effective Yang-Mills field associated to an anomaly such as a monopole and an anyonic statistic in cases of collective motions [14].

Then, to finish with this description, we present the list of magnetic groups compatible with such process of creation of toroidal moments [1] (let us remark that among the 16 compatible groups tabulated in [1], only 12 are associated to a non-trivial $O(2)$ action of the normalizer; that is why four of them, namely the groups 2, 3, 4, 6, are not indicated in the list below):

$$1, 2', m, m', \bar{1}', 2'/m, \bar{3}',$$

$$2/m', 4/m', 6/m', \bar{4}', \bar{6}',$$

(for m and $2'$ we have two TM's oriented in an opposite direction; each one corresponding to possible sublattices). All the corresponding normalizers $N(G)$ contain, as a subgroup, the group of rotations in the plane: $O(2)$, in order to have the precession phenomena. To finish this chapter, let us give a supplementary important remark. The electromagnetic and the “anyonic” potential 4-vectors \mathbf{A} are toroidal moment densities, since they satisfy both the relation (1) in a 3+1 dimensional anyons theory restricted in a 2+1 space, i.e. the fields do not depend on the perpendicular coordinate. Then in fact, the problem is to work out a non-electromagnetic potential vector in magnetoelectric materials associated to each charge carrier. We will see, as a consequence, that CS terms appear and so anyonic statistics.

3 - The relativistic effective Lagrangian

In this chapter, we derive a density of free enthalpy, or rather a relativistic Lagrangian density, since first, we are not interest in thermal properties, and second, because our aim is to obtain a usable “microscopic” Lagrangian in anyons theory. Essentially for simplicity of notations and to avoid discussions about meaning of a lot of particular tensorial coefficients, we will consider a system in a “nearly empty” space since the reasoning we make, will be easily extended in matter without fundamental modifications. The system we consider, is made of a free carrier and only one polarized elementary magnetic cell freely moving in space. In fact, we consider a kind of local crystal field theory.

In some way, the carrier with the cell is a polarized carrier, meaning that the particle gets a covariant (invariant under application of a boost and/or a rotation) tensorial property such as the following scalar product ($\alpha = 0, 1, 2, 3$ and Einstein convention):

$$\mathbf{v} \cdot \mathbf{u} = v^\alpha u_\alpha = cste, \quad (2)$$

where \mathbf{u} is the velocity 4-vector of the carrier (or of the mean position of the carrier in case of an adiabatic process) and \mathbf{v} is a particular 4-vector associated to the magnetic cell. For instance, \mathbf{v} can be the invariant velocity vector of the magnetic symmetry of the cell, or its magnetic or electric polarization vectors. We assume that the interactions between the polarization vectors of the cell and the carrier are only depending on the relative carrier-cell position. As

a fundamental result of the necessary covariance of the relativistic symmetry, any kind of interaction is equivalent to a change of Galilean frame and this is related to the deep meaning of the concept of "polarized carriers". Here, we have a carrier polarized by a cell (!) not only by a spin.

Then, the breaking occurs and the two 4-vectors \mathbf{u} and \mathbf{v} are transformed by a Lorentz transformation $\mathbf{L} \subseteq N(G)$, i.e. we observe in the laboratory frame the primed vectors $\mathbf{u}' = \mathbf{L} \cdot \mathbf{u}$ and $\mathbf{v}' = \mathbf{L} \cdot \mathbf{v}$. Now, deriving relation (2) with respect to the laboratory frame time, we obtain:

$$\dot{\mathbf{v}} \cdot \mathbf{u} + \mathbf{v} \cdot \dot{\mathbf{u}} = 0,$$

where the dot indicates the time derivation. Decomposing $\dot{\mathbf{u}}$ in a colinear and an orthogonal part to \mathbf{v} , we deduce $(\mathbf{a} \times \mathbf{b}) = (a_\alpha b_\beta - a_\beta b_\alpha) E_{\alpha, \beta}$ where $E_{\alpha, \beta}$ is the matrix with 1 on row α and column β and zero everywhere else):

$$\dot{\mathbf{u}} = -(\dot{\mathbf{v}} \cdot \mathbf{u}) \mathbf{v} + \dot{\mathbf{u}}_1 = (\mathbf{v} \times \dot{\mathbf{v}}) \cdot \mathbf{u} + \dot{\mathbf{u}}_1.$$

It can be shown that $\dot{\mathbf{u}}_1$ is only due to different interactions which don't break the symmetry [15]. So, we cancel it out from the latter expression.

This little computation leads us to write:

$$\dot{\mathbf{u}}' = (\dot{\mathbf{L}} \mathbf{L}^{-1} - \mathbf{L} \mathbf{v} \times \mathbf{L} \dot{\mathbf{v}}) \cdot \mathbf{u}' \equiv \mathbf{F}_{eff.} \cdot \mathbf{u}'.$$

This equation is not but the least, a kind of generalized Thomas precession equation [15] which can be associated, in this particular system, to an effective Faraday tensor $\mathbf{F}_{eff.}$ being only a function of the relative cell-carrier position \vec{r} and time, and having not an electromagnetic origin ! But, as a great surprise, the computation of $\mathbf{F}_{eff.}$ shows, first, that its effective electric field is vanishing, and second, its effective magnetic field is such that $(\mathbf{v} \equiv (\gamma, \gamma \vec{v}), \gamma = (1 - \vec{v}^2)^{-1/2}; \theta$ being a function of t and \vec{r}):

$$\vec{B}_{eff.} = \frac{m}{e} \frac{\gamma^2}{(1 + \gamma)} \vec{v} \wedge \dot{\vec{v}} \simeq \theta \vec{v} \wedge (\vec{j}_e \cdot \vec{\nabla}) \vec{v}. \quad (3)$$

We think that it may be possible to interpret this field as the effective magnetic field of the flux tubes leading to the Aharonov–Böhm phase in the anyons theory. We can also notice from the last term in (3) that $\vec{B}_{eff.}$ is a kind of vectorial Lifshitz invariant. From this expression, it follows also that \vec{v} can't be the invariant velocity vector since the breaking doesn't affect its direction. So \vec{v} is, up to a suitable scalar

or pseudoscalar factor, one of the two polarization vectors of the magnetic cell. Thus, if for instance m' is the magnetic symmetry, since the two latter vectors are in the plane of the symmetry, then $\vec{B}_{eff.}$ is in the orthogonal direction. We have also to notice, as a consequence of the model, that there is such an effective magnetic field (and so a toroidal moment) associated to each charge carrier.

At this step, unfortunately, we need very sophisticated mathematical tools such as exterior differentiation d , co-differentiation δ , exact, co-exact and harmonic differential n-forms, coming from the de Rham cohomology theory of differential manifolds (see [9] for instance).

The fundamental consequence of the previous relation is that the total Faraday tensor $\mathbf{F}_{Tot.} = \mathbf{F} + \mathbf{F}_{eff.}$ is no longer a closed form, i.e. $d\mathbf{F}_{Tot.} \neq 0$ ($\iff \text{div} \vec{B}_{eff.} \neq 0$) because of the time t and the vector position \vec{r} dependencies. It follows from the Hodge duality principle that we can define $d\mathbf{F}_{Tot.}$ ($\delta\mathbf{F}_{Tot.}$) as a current of an "effective" magnetic (electric) charge: $d\mathbf{F}_{Tot.} = * \mathbf{j}_m$ ($\delta\mathbf{F}_{Tot.} = \mathbf{j}_e$). Let us remark that the currents are not primed, because they are currents in the laboratory frame, or equivalently, they have zero divergences in contradistinction with the primed currents. Moreover, it means also that the electric charge-matter interaction is equivalent to an electric charge-magnetic charge interaction.

Hence, if the Lagrangian density for the "electromagnetic" Faraday tensor \mathbf{F} is (with Heaviside's units):

$$-\frac{1}{4} \mathbf{F} \cdot \mathbf{F} - \mathbf{A} \cdot \mathbf{j},$$

we obtain for $\mathbf{F}_{Tot.}$ an other Lagrangian density \mathcal{L} such that:

$$\mathcal{L} \equiv -\frac{1}{4} \mathbf{F}_{Tot.} \cdot \mathbf{F}_{Tot.} - \mathbf{A}_e \cdot \mathbf{j}_e - \frac{\kappa}{4} * \mathbf{F}_{Tot.} \cdot \mathbf{F}_{Tot.} - \kappa \mathbf{A}_m \cdot \mathbf{j}_m, \quad (4)$$

where κ is a constant, $* \mathbf{F}_{Tot.}$ the Hodge dual of $\mathbf{F}_{Tot.}$, and \mathbf{A}_e and \mathbf{A}_m are respectively the potential 4-vectors associated to the exact part of $\mathbf{F}_{Tot.}$ and $* \mathbf{F}_{Tot.}$. As for the currents, the \mathbf{A} 's are not primed. Let us remark that the harmonic parts have no contributions to the dynamic of the charges, since they differentiations and co-differentiations are vanishing, but they have one in the energy and consequently in the Lagrangian density. These parts are associated

to the so-called instantons which are fields generated by the breaking and going away from the breaking area, carrying out a part of the energy. Nevertheless, in the case of more than one electric charge, this field will interact with the others and can not be neglected.

Thus, the total potential 4-vector \mathbf{A} , interacting with a one-charge current, will be the sum of the potentials of all the closed parts of the total corresponding Faraday tensor, for which the Poincaré's lemma can be applied (indeed, from this lemma, in a suitable open subset not containing the interaction area, it always exists a potential since there is no more closed part inside). In fact, because $\mathbf{F}_{Tot.}$ is not only electromagnetic and

$$\delta\mathbf{F}_{Tot.} = \mathbf{j}_e \implies \text{curl}(\text{curl}(\vec{A}_e)) \stackrel{\text{def.}}{=} \vec{j}_{\mathbf{A}_e}$$

$$d\mathbf{F}_{Tot.} = {}^*\mathbf{j}_m \implies \text{curl}(\text{curl}(\vec{A}_m)) \stackrel{\text{def.}}{=} \vec{j}_{\mathbf{A}_m},$$

then \vec{A}_e and \vec{A}_m are toroidal moment densities: $\vec{A}_{e,m} \equiv \vec{T}_{e,m}$. Moreover, ${}^*\mathbf{F}_{Tot.} \cdot \mathbf{F}_{Tot.}$ is the expected Chern-Simon term of anyons theory and the harmonic part of $\mathbf{F}_{Tot.}$ will correspond to the famous singular potential of this theory, i.e. to magnetic monopoles. Consequently, at a macroscopic level, we must add in the free enthalpy, terms such as (E : electric field, H : magnetic field, J : current, T : dipolar toroidal moment):

$$\gamma_{i,j} E^i J_{e,m}^j, \quad \mu_{i,j} H^i J_{e,m}^j,$$

$$\rho_{i,j} J_{e,m}^i J_{e,m}^j, \quad \lambda_{i,j} J_e^i J_m^j,$$

terms, up to order four, coming from relations (3) and (4), analogous terms substituting T^i for J^i , and crossing terms such as

$$\tau_{i,j} J_{e,m}^i T_{e,m}^j, \quad \omega_{i,j} J_{e,m}^i T_{m,e}^j,$$

where γ , μ , ρ , λ , τ and ω are the corresponding suitable susceptibility tensors. To finish, let us add that it might allow permanent currents for particular choices of the latter tensors, and so superconductivity with toroidal phase transitions [12].

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